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SHORTER COMMUNICATION

COMPARISON OF METHODS FOR SOLUTION OF THE HEAT CONDUCTION EQUATION WITH A RADIATION BOUNDARY CONDITION

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NOMENCLATURE

C, specific heat of slab;
k, thermal conductivity of slab;
L, slab thickness;
M, $(\Delta X)^2/\Delta\tau$;
N, $\epsilon\sigma L T_i^3/k$;
t, time;
T, slab temperature (absolute);
T_i, initial slab temperature (absolute);
x, distance;
X, x/L.

Greek Symbols
ΔX, incremental step length in X-direction;
$\Delta\tau$, incremental step length in τ-direction;
ϵ, emissivity of slab surface;
θ, T/T_i;
ρ, density of slab;
σ, Stefan–Boltzmann constant;
τ, $kt/\rho CL^2$.

INTRODUCTION

TRANSIENT heat conduction with radiation boundary conditions arises in a range of heat-transfer problems. Since such problems are essentially non-linear, numerical or other approximate methods of solution must be used. The object of this note is to evaluate the suitability of a number of approximate methods by applying them to the same problem.

THE PROBLEM

The problem discussed is that previously considered by Fairall *et al.* [1] and Schneider [2, 3]. A slab of finite thickness is considered, with radiation at one face to a sink at absolute zero temperature and perfect insulation at the other. In non-dimensional form the relevant equations are

$$\frac{\partial\theta}{\partial\tau} = \frac{\partial^2\theta}{\partial X^2}$$

$$X = 0 \quad \frac{\partial\theta}{\partial X} = N\theta^4$$

$$X = 1 \quad \frac{\partial\theta}{\partial X} = 0$$

$$\tau = 0 \quad \theta = 1$$

This problem, although somewhat idealized, represents certain physical situations [1] and will provide a basis for comparing the various methods of solution.

METHODS OF SOLUTION

Fairall *et al.* [1] have previously used an explicit finite difference technique proposed by Dusinberre [4]. In a similar problem Schneider [2, 3] has employed an integral method assuming the temperature distribution throughout the slab to be a quadratic function of distance. In the present work the implicit Crank–Nicolson finite difference method [5] and analogue computation have been used. In addition an integral solution employing a cubic temperature profile has been obtained and the results given by Fairall and Schneider have been recomputed for comparison purposes.

RESULTS

All numerical calculations were performed on a National Elliott 803 digital computer. The explicit method was applied with values of $\Delta X = 0{\cdot}1$ and $0{\cdot}05$. The results obtained were in reasonable agreement with those of Fairall *et al.* although convergence was not complete particularly at $X = 1$. Further calculations with smaller values of ΔX would have been desirable but the computations were rather time-consuming and were not continued. This feature of the explicit method is due to the small values of $\Delta\tau$ required for stability of the solution.

The implicit method was used with values of ΔX of 0·1, 0·05 and 0·025 and values of $\Delta\tau$ of 0·02, 0·01 and 0·005. Inspection of the results indicated that convergence was virtually complete for $\Delta X = 0{\cdot}1$ and $\Delta\tau = 0{\cdot}01$ although it is believed that large values of the radiation parameter N would require a smaller value of $\Delta\tau$. Despite its iterative nature, the implicit method was considerably faster than the explicit method due to the choice of $\Delta\tau$ not being limited by stability considerations. However, the modulus M should not be too small (say $\geqslant 1$) otherwise the solution oscillates for small values of τ.

Due to limitations of equipment the analogue computations were carried out with $\Delta X = 0{\cdot}2$. The solutions were slowed down by a factor of 25 in order for them to be compatible with the recording equipment.

In general, the discrepancies between the various methods are greater for large N and as $X \to 1$. Typical results are shown in Fig. 1 for $N = 10$ and $X = 1$ and

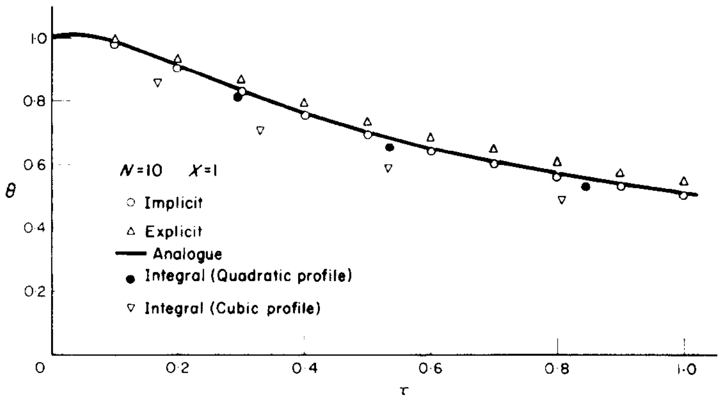

FIG. 1.

were obtained by analogue computations ($\Delta X = 0{\cdot}2$), the explicit method ($\Delta X = 0{\cdot}1$), the implicit method ($\Delta X = 0{\cdot}1$, $\Delta\tau = 0{\cdot}01$) and the integral method (quadratic and cubic profiles).

The explicit method converges rather slowly and, since it is time-consuming, is inferior to the implicit method. The integral solution employing a quadratic profile also gives satisfactory results but the solution based on a cubic profile is substantially in error. This problem appears to be one of the cases where increasing the degree of the polynomial used decreases the accuracy of the solution. Goodman [6] remarks upon this possibility. It should be noted that the integral solution (quadratic profile) given in Fig. 1. has been re-calculated by the author from the equations given in reference 2. The curves given in reference 3 (Chart 52) appear to be in error particularly in the region $0{\cdot}1 < \tau < 1{\cdot}0$.

CONCLUSIONS

The implicit Crank–Nicolson method is recommended for digital computation. The explicit Dusinberre method converges rather gradually and is relatively slow because of the limitation imposed on $\Delta\tau$ by stability considerations. The analogue computation technique is satisfactory even for a rather large value of ΔX. The integral solution (quadratic profile) is satisfactory and rapid to use but the cubic profile solution is not recommended.

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